I. Settings

Fix p prime. $K \mid \mathbb{Q}_p$ finite extension. It is thus a completion of a number field. $\exists$ an absolute value on K extending $|\cdot|_p$. Let $\mathcal{O}_K$ be its ring of integers. $\mathfrak{m}_K$ = unique max ideal.

$$\mathcal{O}_K = \{ x \in F : |x|_K \leq 1 \}.$$

$$\mathfrak{m}_K = \{ x \in F : |x|_K < 1 \}.$$

Write $k = \mathcal{O}_K / \mathfrak{m}_K$ be the finite residue field.

$E_K(z) \in \mathcal{O}_K[z]$ be the Eisenstein poly.

Fix π_K a uniformizer of K. Now $[K : \mathbb{Q}_p] = [k : \mathbb{F}_p] \cdot e_K$. where e_K = ramification index ($p\mathcal{O}_K = \mathfrak{m}_K^{e_K}$).

Consider $\mathcal{O}_K$ as an E_∞-alg over $\mathbb{S}_{W(k)}[z]$ via

$$\mathbb{S}_{W(k)}[z] \longrightarrow W(k)[z] \longrightarrow \mathcal{O}_K.$$

$$z \longmapsto \pi_K$$

where $\mathbb{S}_{W(k)}$ is the spherical Witt vectors. which are a lift of Witt vectors from $\mathbb{Z}_p$ to $\mathbb{S}_p^\wedge$. p-complete sphere spectrum.

- <u>Thm</u> (Nikolaus - Yakerson. 2024)

  $$\mathbb{S}_{W(k)} \simeq \mathbb{S}[k]_I^\wedge = \varprojlim \mathbb{S}[k] / I^n.$$

  as E_∞-rings. where $I = \mathrm{fib}(\mathbb{S}[k] \longrightarrow k)$. $I^n = I^{\otimes_A n}$.

- Original definition of $\mathbb{S}_{W(k)}$:

<u>Thm</u> (Lurie)

In $\mathrm{CommAlg}(\mathrm{Sp})$. consider $\mathbb{S}/(p)$ (or $\mathbb{S}_p^\wedge/(p)$). then

$\pi_0(\mathbb{S})/(p) = \mathbb{F}_p$ (or $\pi_0(\mathbb{S}_p^\wedge)/(p) = \mathbb{F}_p$). Suppose $f_0 : \pi_0\mathbb{S}/(p) \longrightarrow B_0$ is relatively perfect, or equivalently B_0 is a perfect $\mathbb{F}_p$-alg. Then we have a lift of B_0 to a flat $\mathbb{S}$-algebra B, s.t. it is complete w.r.t. (p) and $\mathbb{F}_p \otimes_{\mathbb{S}} B \simeq \pi_0(B)/p\pi_0 B \simeq B_0$.

$$\begin{tikzcd}
\mathbb{S} \arrow[r, "f"] \arrow[d] & B \arrow[d] \\
\mathbb{S}/(p) \arrow[r, "f_0"] & B_0
\end{tikzcd}$$

Now this B is the spherical Witt vectors $\mathbb{S}_{W(B_0)}$.

Lastly. $\mathbb{S}_{W(k)}[z] := \mathbb{S}_{W(k)} \otimes_{\mathbb{S}} \mathbb{S}[z] = \mathbb{S}_{W(k)} \otimes_{\mathbb{S}} \mathbb{S}[\mathbb{N}]$

$$= \mathbb{S}_{W(k)} \otimes_{\mathbb{S}} \Sigma^\infty_+ \mathbb{N}$$

is the free E_∞-ring spectrum gen. by the comm. monoid $\mathbb{N}$.

$\pi_* \mathbb{S}_{W(k)}[z] = (\pi_* \mathbb{S}_{W(k)})[z]$. $\pi_0 \mathbb{S}_{W(k)} = W(k)$ flat over $\mathbb{S}/(p)$.

II. Problem

★ Compute $TC_*(\mathcal{O}_K)$ and $TC^-_*(\mathcal{O}_K)$. $TP_*(\mathcal{O}_K)$.

III. Strategy

1. Adams resolution :

$$\mathbb{S}_{W(k)} \longrightarrow \mathbb{S}_{W(k)}[z]^{\otimes[-]}$$

$$\rightsquigarrow \quad THH(\mathcal{O}_K/\mathbb{S}_{W(k)}) \longrightarrow THH(\mathcal{O}_K/\mathbb{S}_{W(k)}[z]^{\otimes[-]})$$

cosimplicial E_∞-alg in cyclotomic spectra.

$\rightsquigarrow$ Similar for TC^- and TP.

multiplicative property $\rightsquigarrow$

$$THH(\mathcal{O}_K/\mathbb{S}_{W(k)}) \to THH(\mathcal{O}_K/\mathbb{S}_{W(k)}[z])^{\otimes[n]}.$$

TC^-, TP.

By coskeleton filtrations, get multiplicative 2^{nd} quadrant homology type spectral sequence, a.k.a. descent spectral sequences:

$$E^1_{s,t}(THH(\mathcal{O}_K)) = THH_t(\mathcal{O}_K/\mathbb{S}_{W(k)}[z]^{\otimes[-s]})$$
$$\Rightarrow THH_{s+t}(\mathcal{O}_K/\mathbb{S}_{W(k)})$$

Similarly for TC^- and TP.

2. Hopf algebroid:

$$(THH_*(\mathcal{O}_K/\mathbb{S}_{W(k)}[z]),\ THH_*(\mathcal{O}_K/\mathbb{S}_{W(k)}[z_0, z_1]))$$

is a Hopf algebroid. It follows that E^1-term of descent SS for THH is the cobar cpx for $THH_*(\mathcal{O}_K/\mathbb{S}_{W(k)}[z])$ w.r.t. the Hopf algebroid. Now

$$E^2_{s,t} = \mathrm{Ext}^{-s,t}_{THH_*(\mathcal{O}_K/\mathbb{S}_{W(k)}[z_0, z_1])}(THH_*(\mathcal{O}_K/\mathbb{S}_{W(k)}[z]))$$

and for TP

$$E^2_{s,t} = \mathrm{Ext}^{-s}_{TP_0(\mathcal{O}_K/\mathbb{S}_{W(k)}[z_0, z_1])}(TP_t(\mathcal{O}_K/\mathbb{S}_{W(k)}[z]))$$

TC^- is a little different, but not essential.

3. Compute Hopf algebroid:

Use δ-ring str on $TP_0(\mathcal{O}_K/\mathbb{S}_{W(k)}[z])$ given by cyclotomic Frobenius φ to determine

$$TP_0(\mathcal{O}_K/\mathbb{S}_{W(k)}[z_0, z_1]).$$

and use an invariant of HKR theorem to compute

$$THH_*(\mathcal{O}_K/\mathbb{S}_{W(k)}[z_0, z_1]) = \mathcal{O}_K[u_0] \otimes_{\mathcal{O}_K} \mathcal{O}_K\langle t_{z_0 - z_1}\rangle$$

where $$I/I^2 \cong HH_2(\cdots) \cong THH_2(\cdots)$$

$$z_0 - z_1 \longmapsto t_{z_0 - z_1}$$

for $$I = \ker(W(k)[z_0, z_1] \xrightarrow{z_0, z_1 \to \pi_K} \mathcal{O}_K).$$

Use these info to compute E^2-term of the descent SS for THH.

4. Algebraic Tate spectral sequences :

Direct computation of E^2-term of TP, TC^- in the descent SS is hard. But note that $E^1(TP(\mathcal{O}_K))$ endowed w/ the Nygaard filtration (inherited from the one on $TP(\mathcal{O}_K/\mathbb{S}_{W(k)}[z]^{\otimes[-]})$)

Now assoc. graded $= E^1(THH(\mathcal{O}_K))[\sigma^{\pm}]$

b/c Tate SS for $TP(\mathcal{O}_K/\mathbb{S}_{W(k)}[z]^{\otimes[-]})$ collapse at E^2.

Then the "algebraic Tate SS" goes like

$$E^2(THH(\mathcal{O}_K))[\sigma^{\pm}] \Rightarrow E^2(TP(\mathcal{O}_K))$$

$$E^2(\quad \cdots \quad)[v] \Rightarrow E^2(TC^-(\mathcal{O}_K))$$

while the later a.k.a. "algebraic HFPSS".

5. Summarize :

from Nygaard filtration

from cosimplicial filtration

$$\begin{tikzcd}[column sep=small]
 & \begin{matrix} TC^- \\ E^2(TP(\mathcal{O}_K)) \end{matrix} \arrow[dr, Rightarrow, "\text{descent SS}"] & \\
\underbrace{E^2(THH(\mathcal{O}_K))}_{\text{computed by cobar complex}}[\cdot\cdot] \arrow[ur, Rightarrow, "\text{algebraic Tate SS}"] \arrow[dr, Rightarrow, "\text{descent SS}"'] & & \begin{matrix} TC^-_* \\ TP_*(\mathcal{O}_K/\mathbb{S}_{W(k)}) \end{matrix} \\
 & THH(\mathcal{O}_K)[\cdot\cdot] \arrow[ur, Rightarrow, "\text{Tate SS}"'] &
\end{tikzcd}$$

from cosimplicial filtration

collapse at E^2

6. TC part :

Start from

$$can.\ \varphi : TC^-(\mathcal{O}_K/\mathbb{S}_{W(k)}[z]^{\otimes[-]}) \to TP(\mathcal{O}_K/\mathbb{S}_{W(k)}[z]^{\otimes[-]})$$

Define $TC(\mathcal{O}_K/\mathbb{S}_{W(k)})_{(n)}$ = fiber of

$$can - \varphi : \lim_{\Delta^{\le n}} TC^-(\cdot\cdot) \longrightarrow \lim_{\Delta^{\le n-1}} TP(\cdot\cdot)$$

and

$$TC(\cdot\cdot)_{(n)} / TC(\cdot\cdot)_{(n+1)} \simeq \frac{\lim_{\Delta^{\le n}} TC^-(\cdot\cdot)}{\lim_{\Delta^{\le n+1}} TC^-(\cdot\cdot)} \oplus \Sigma^{-1} \frac{\lim_{\Delta^{\le n-1}} TP(\cdot\cdot)}{\lim_{\Delta^{\le n}} TP(\cdot\cdot)}$$

The tower $\{TC(\cdot\cdot)_{(n)}\}_{n\ge 0}$ gives the descent SS :

$$E^1_{s,t}(TC(\mathcal{O}_K)) \Rightarrow TC_{s+t}(\mathcal{O}_K/\mathbb{S}_{W(k)})$$

$$\tilde{E}^2_{s,u,t}(TC(\mathcal{O}_K)) \Rightarrow E^2_{s-u,t}(TC(\mathcal{O}_K)) \qquad u \in \{0,1\}.$$

where

$$\tilde{E}^2_{s,0,t} = \ker(can - \varphi : E^2_{s,t}(TC^-(\mathcal{O}_K)) \to E^2_{s,t}(TP(\mathcal{O}_K)))$$

$$\tilde{E}^2_{s,1,t} = \operatorname{coker}(can - \varphi : E^2_{s,t}(TC^-(\mathcal{O}_K)) \to E^2_{s,t}(TP(\mathcal{O}_K)))$$

IV. Outline and sketch

▲ FACT

1) $THH_*(\mathcal{O}_K / \mathbb{S}_{W(k)}[z]) = \mathcal{O}_K[u]$

where $u \in THH_2(\mathcal{O}_K / \mathbb{S}_{W(k)}[z])$ lift of Bökstedt elet in $THH_2(k)$.

2) Tate SS for $TP_*(\mathcal{O}_K/\mathbb{S}_{W(k)}[z])$ collapse at E^2-term.

$$TP_*(\mathcal{O}_K/\mathbb{S}_{W(k)}[z]) = TP_0(\mathcal{O}_K/\mathbb{S}_{W(k)}[z])[\sigma^{\pm}]$$

where $|\sigma| = 2$.

3) $TP_0(\mathcal{O}_K/\mathbb{S}_{W(k)}[z]) = W(k)[\![z]\!]$

and

$$\begin{tikzcd}
p_0 : TP_0(\mathcal{O}_K/\mathbb{S}_{W(k)}[z]) \arrow[r] \arrow[d, equal] & THH_0(\mathcal{O}_K/\mathbb{S}_{W(k)}[z]) \arrow[d, equal] \\
W(k)[\![z]\!] \arrow[r] & \mathcal{O}_K \\
z \arrow[r, mapsto] & \pi_K
\end{tikzcd}$$

is a $W(k)$-alg morphism.

4) HFPSS for $TC^-_*(\mathcal{O}_K/\mathbb{S}_{W(k)}[z])$ collapse at E^2-term.

$$\mathrm{can} : TC^-_j(\mathcal{O}_K/\mathbb{S}_{W(k)}[z]) \longrightarrow TP_j(\mathcal{O}_K/\mathbb{S}_{W(k)}[z])$$

induces iso $TC^-_j(\mathcal{O}_K/\mathbb{S}_{W(k)}[z]) \cong \mathcal{N}^{\geq j} TP_j(\mathcal{O}_K/\mathbb{S}_{W(k)}[z])$ for all $j \in \mathbb{Z}$. "can" is actually an iso for $j \leq 0$.

5)

$$\begin{tikzcd}
\varphi : TC^-_0(\mathcal{O}_K/\mathbb{S}_{W(k)}[z]) \arrow[r] \arrow[d, equal] & TP_0(\mathcal{O}_K/\mathbb{S}_{W(k)}[z]) \arrow[d, equal] \\
W(k)[\![z]\!] \arrow[r] & W(k)[\![z]\!] \\
z \arrow[r, mapsto] & z^p
\end{tikzcd}$$

is a Frobenius on $W(k)$.

6) Let $u_{\mathbb{F}_p} \in TC_2^-(\mathbb{F}_p)$ lifting the Bökstedt elet under p_0 in the $\mathbb{F}_p$ case. Then $\exists!$ $\tilde{u} \in TC_2^-(\mathcal{O}_K/\mathbb{S}_{W(k)}[z])$

$v \in TC_{-2}^-(\mathcal{O}_K/\mathbb{S}_{W(k)}[z])$

$\sigma \in TP_2(\mathcal{O}_K/\mathbb{S}_{W(k)}[z])$

s.t. $\tilde{u}$ lifts $u_{\mathbb{F}_p}$

$\varphi(\tilde{u}) = \sigma$

$\mathrm{can}(v) = \sigma^{-1}$

and
$$TC_*^-(\mathcal{O}_K/\mathbb{S}_{W(k)}[z]) = \frac{TC_0^-(\mathcal{O}_K/\mathbb{S}_{W(k)}[z])[\tilde{u}.v]}{\tilde{u}v - E_K(z)}$$

7) Nygaard filtration on $TP_0(\mathcal{O}_K/\mathbb{S}_{W(k)}[z])$ is by
$$\mathcal{N}^{\geq 2j} TP_0(\mathcal{O}_K/\mathbb{S}_{W(k)}[z]) = \mathcal{N}^{\geq 2j-1} TP_0(\mathcal{O}_K/\mathbb{S}_{W(k)}[z])$$
$$= (E_K(z))^j \quad , \quad j \geq 0.$$

From now on, regard $\mathcal{O}_K$ as $\mathbb{S}_{W(k)}[z]^{\otimes[-]}$-alg via the map
$$\mathbb{S}_{W(k)}[z]^{\otimes[-]} \xrightarrow[\forall i \geq 0]{z_i \mapsto \pi_K} \mathcal{O}_K$$

Recall in the summary. we need :

Step 2 from Nygaard filtration

Step 3 algebraic Tate SS

from cosimplicial filtration

descent SS

$$\begin{tikzcd}
 & TC^- \\
 & E^2(TP(\mathcal{O}_K)) \arrow[dr, Rightarrow] & \\
E^2(THH(\mathcal{O}_K))[\cdot\cdot] \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & TC_*^- \\
 & & TP_*(\mathcal{O}_K/\mathbb{S}_{W(k)}) \\
 & THH(\mathcal{O}_K)[\cdot\cdot] \arrow[ur, Rightarrow] &
\end{tikzcd}$$

computed by

descent

Tate SS

cobar complex   SS   from cosimplicial filtration   collapse at E^2

Step 1

- <u>Step 1</u> : Compute $E^2(\mathrm{THH}(\mathcal{O}_K))$

  As graded rings, we have

$$\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(k)}[z_0, z_1]) = \mathcal{O}_K[u_0] \otimes_{\mathcal{O}_K} \mathcal{O}_K\langle t_{z_0-z_1}\rangle$$

  This actually follows from the Nygaard filtration on $\mathrm{TP}_0(\cdot\cdot)$.

  We will get to there at the end of this section.

  Here $\quad z_0 = \eta_L(z) \qquad u_0 = \eta_L(u)$

  $\qquad\quad z_1 = \eta_R(z) \qquad u_1 = \eta_R(u)$

  for η_L, η_R left & right units.

<u>Prop</u> $\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(k)}[z]^{\otimes[-]})$ is a Hopf algebroid obj in the cat of graded rings.

The Hopf algebroid str on

$$(\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(k)}[z]),\ \mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(k)}[z_0, z_1]))$$

$$= (\mathcal{O}_K[u],\ \mathcal{O}_K[u_0] \otimes_{\mathcal{O}_K} \mathcal{O}_K\langle t_{z_0-z_1}\rangle)$$

$$\eta_L : u \longmapsto u_0$$

$$\eta_R : u \longmapsto u_1 = u_0 - E_K'(\pi_K)\, t_{z_0-z_1}$$

$$\Delta(t^{[i]}_{z_0-z_1}) = \sum_{0\le j\le i} t^{[j]}_{z_0-z_1} \otimes t^{[i-j]}_{z_0-z_1}$$

$$\varepsilon(t_{z_0-z_1}) = 0, \quad \varepsilon : u_0, u_1 \longmapsto u.$$

$$c(u_0) = u_1, \quad c(u_1) = u_0, \quad c(t_{z_0-z_1}) = t_{z_1-z_0}.$$

Recall the E^2-page of descent SS for THH :

$$E^2_{s,t} = \mathrm{Ext}^{-s,t}_{THH_*(\mathcal{O}_K/\mathbb{S}_{W(k)}[z_0,z_1])}(THH_*(\mathcal{O}_K/\mathbb{S}_{W(k)}[z]))$$

The injective resolution of $THH_*(\mathcal{O}_K/\mathbb{S}_{W(k)}[z])$ as left $THH_*(\mathcal{O}_K/\mathbb{S}_{W(k)}[z_0,z_1])$-modules goes like

$$0 \to THH(\mathcal{O}_K/\mathbb{S}_{W(k)}[z]) \xrightarrow{\eta_L} THH_*(\mathcal{O}_K/\mathbb{S}_{W(k)}[z_0,z_1])$$
$$\xrightarrow{x \mapsto D(x)dz} THH_*(\mathcal{O}_K/\mathbb{S}_{W(k)}[z_0,z_1])\,dz \longrightarrow 0$$

where $D: THH_*(\mathcal{O}_K/\mathbb{S}_{W(k)}[z_0,z_1]) \longrightarrow THH_*(\mathcal{O}_K/\mathbb{S}_{W(k)}[z_0,z_1])$

$$t^{[i]}_{z_0-z_1} \longmapsto t^{[i-1]}_{z_0-z_1}$$

and $|dz| = 2$

Thus taking $\mathrm{Hom}_{THH_*(\mathcal{O}_K/\mathbb{S}_{W(k)}[z_0,z_1])}(THH_*(\mathcal{O}_K/\mathbb{S}_{W(k)}[z]),-)$ gives (Write $A = THH_*(\mathcal{O}_K/\mathbb{S}_{W(k)}[z])$

$\Gamma = THH_*(\mathcal{O}_K/\mathbb{S}_{W(k)}[z_0,z_1])$ for simplicity).

$$\mathrm{Hom}_\Gamma(A,\Gamma) \xrightarrow{\mathrm{Hom}_\Gamma(A,-)\circ D(-)dz} \mathrm{Hom}_\Gamma(A,\Gamma)\,dz \qquad (*)$$

<u>FACT</u> (A,Γ) Hopf algebroid. M left Γ-mod. N A-mod.

$$\Rightarrow \quad \mathrm{Hom}_A(M,N) \cong \mathrm{Hom}_\Gamma(M,\Gamma\otimes_A N)$$
$$f \longmapsto (id\otimes f)\circ\Delta =: \tilde{f}.$$

Apply the fact to $(*)$, where $A = M = N$. then have

$$\mathrm{Hom}_A(A,A) \xrightarrow{f \mapsto (D\circ\tilde{f})dz} \mathrm{Hom}_A(A,A)\,dz$$

$$\rightsquigarrow \quad \mathrm{Hom}_A(A,A) \cong A \quad . \quad (D\circ\tilde{f})dz = (D_0\circ\eta_R)\,dz$$
$$f \longmapsto f(1)$$

and $D_0 : \Gamma \longrightarrow A$ map of left A-mod by

$$t_{z_0 - z_1}^{[i]} \longmapsto \begin{cases} 1 , & i = 1 \\ 0 , & \text{else} \end{cases}$$

$\rightsquigarrow$ $\mathrm{Ext}_\Gamma^{0,0}(A) \cong \mathcal{O}_K$

$\mathrm{Ext}_\Gamma^{1,2n}(A) \cong \mathcal{O}_K / (n E_K'(\pi_K))$, $n \geq 1$

other vanishes. So the descent SS collapse at E^2-page by degree reason.

$$\rightsquigarrow \mathrm{THH}_*(\mathcal{O}_K / \mathbb{S}_{W(k)}[z]) = \begin{cases} \mathcal{O}_K , & * = 0 \\ \mathcal{O}_K / n E_K'(\pi_K) , & * = 2n-1 \\ 0 , & \text{else} \end{cases}$$

where $n \geq 1$.

- <u>Step 2</u> : Nygaard filtration on TP_0

From the fact, we already know $\mathcal{N}^{\geq j}\,\mathrm{TP}_0(\mathcal{O}_K / \mathbb{S}_{W(k)}[z])$

We need to know the Nyg. filtration for $\mathrm{TP}_0(\mathcal{O}_K / \mathbb{S}_{W(k)}[z_0, z_1])$.

<u>Aside</u> Divided power ring consists of the following data:

(A, I, γ), A ring, $I \subseteq A$ ideal,

$\gamma = (\gamma_n)_{n \geq 0}$ divided power structure on I, i.e. $\gamma_n : I \longrightarrow A$

s.t. 1) $\gamma_0(x) = 1$, $\gamma_1(x) = x$, $\gamma_n(x) \in I$

2) $\gamma_n(x+y) = \sum_{i=0}^{n} \gamma_i(y)\, \gamma_{n-i}(x)$

3) $\gamma_n(\lambda x) = \lambda^n \gamma_n(x)$, $\forall \lambda \in A$

4) $\gamma_m(x)\gamma_n(x) = \frac{(m+n)!}{m!\,n!}\gamma_{m+n}(x)$

5) $\gamma_n(\gamma_m(x)) = \frac{(mn)!}{(m!)^n n!}\gamma_{mn}(x)$. $m>0$

Write $\gamma_n(x) = x^{[n]}$.

The divided power envelope is the universal obj $D_B(J)$

s.t. $\mathrm{Hom}_{\text{Divided Ring}}(D_B(J), (C, K, \delta))$

$= \mathrm{Hom}_{\text{Ring}}((B, J), (C, K))$

for (B, J, γ).

<u>Thm</u> (HKR)

R comm. ring / $\mathbb{Z}_p$. I locally complete intersection ideal of R. $A = R/I$ p-torsion free. Assume R is I-separated. then

1) $HP_0(A/R) \cong D_R(I)^\wedge_{\mathcal{N}}$ w.r.t. Nyg. fil.

2) $HH_*(A/R) \cong \Gamma_A(I/I^2)$, where Γ_A = divided power alg.

Now apply HKR to $R = W(k)[z_0, z_1]$

$I = \ker(W(k)[z_0, z_1] \xrightarrow{z_0, z_1 \mapsto \pi_K} \mathcal{O}_K)$

$A = \mathcal{O}_K$.

$\Rightarrow I/I^2 \cong HH_2(A/R) \cong THH_2(A/\mathbb{S}_R)$

$z_0 - z_1 \longmapsto t z_0 - z_1$.

Prop The graded assoc. to Nyg. fil. of $TP_0(\cdot\cdot)$ is $THH_*(\cdot\cdot)$.

In order to understand $TP_0(\mathcal{O}_K/\mathfrak{S}_{W(k)}[z_0, z_1])$, we need two things:

1) $\mathcal{N}$-top and $(p, \mathcal{N})$-top.
2) δ-ring str.

To summarize:

1) **Def** R w/ multiplicative decreasing filtration (e.g. Nyg. filtration). $\mathcal{N}$-top is the topology by this filtration $\mathcal{N}^{\geq \bullet}$. $(p, \mathcal{N})$-top is the topology in which $\{(p^j) + \mathcal{N}^{\geq j}\}_{j\geq 0}$ forms a basis of open nbh of 0. Both are NOT adic top!

Prop $TP_0(\mathcal{O}_K/\mathfrak{S}_{W(k)}[z_0, z_1])$ is complete & separated in both $\mathcal{N}$-top and $(p, \mathcal{N})$-top, and cyclotomic Frob. is cts in $(p, \mathcal{N})$-top.

2) **Def** A δ-ring is a pair (R, δ). R comm. ring $\delta: R \longrightarrow R$ map of sets w/ $\delta(0) = \delta(1) = 0$, and

$$\delta(xy) = x\delta(y) + y\delta(x) + p\delta(x)\delta(y)$$
$$\delta(x+y) = \delta(x) + \delta(y) + \frac{1}{p}(x^p + y^p - (x+y)^p).$$

If R p-torsion free, δ-ring str. on R is equiv. to

$\varphi : R \longrightarrow R$ lifting the Frob. on R/p. Explicitly.

$$\delta(x) = \frac{\varphi(x) - x^p}{p}.$$

<u>Prop</u> The cyclotomic Frob. on $TP_0(\cdot\cdot)$ makes it a δ-ring.

Last. $TP_0(\mathcal{O}_K/\mathbb{S}_{W(k)}[z_0.z_1]) \cong$ closure of subring of

$D_{W(k)[z_0,z_1]}(E_K(z_0).\ z_0 - z_1)^{\wedge}_{\mathcal{N}}$

gen. by $W(k)[z_0.z_1]$ and

$\{\iota(\delta^k(h))\}_{k\geq 0}$ under either top.

Here $\iota.\ \delta.\ h$ from δ-ring str. on $TP_0(\cdot\cdot)$:

$$\delta^0 h = h. \quad \delta^{k+1} h = \tfrac{1}{p}(\varphi(\delta^k(h)) - \delta^k(h)^p)$$

$$f^{(0)} = z_0 - z_1, \quad f^{(k+1)} = \tfrac{1}{p}(-(f^{(k)})^p + \delta^k h \cdot E_K(z_0)^{p^{k+1}})$$

$$h\varphi(E_K(z_0)) = \varphi(z_0 - z_1) = z_0^p - z_1^p$$

s.t.

$$\delta^k h \cdot \varphi(E_K(z_0))^{p^k} = \varphi(f^{(k)})$$

$$\delta^k h \in TP_0(\cdot\cdot)$$

$$f^{(k)} \in \mathcal{N}^{\geq 2p^k} TP_0(\cdot\cdot)$$

ι is inclusion.

<u>Prop</u> $TP_0(\mathcal{O}_K/\mathbb{S}_{W(k)}[z]^{\otimes[-]})$ a Hopf algebroid in the cat of complete filtered rings. $\delta^k h \in \mathcal{N}^{\geq 2} TP_0(\mathcal{O}_K/\mathbb{S}_{W(k)}[z_0.z_1])$.

- **Step 3**: Algebraic Tate SS and E^2-term of descent SS.

By Hopf algebroid, we have E^2-page by cobar cpx.

$$E^2_{s,t} = \mathrm{Ext}^{-s}_{TP_0(O_K/\mathbb{S}_{W(k)}[z_0, z_1])}(TP_t(O_K/\mathbb{S}_{W(k)}[z]))$$

But this is hard. The structure of it is complicated.

Instead, first note

Def The Nyg. filtration $\mathcal{N}^{\geq j}$ is defined on $E^\infty_{*,j}$ of Tate SS $E^2_{i,j} = THH_*(-/E)[\sigma^\pm] \Rightarrow TP_{i+j}(-/E)$.

If it collapse at E^2, then $\mathcal{N}^{\geq j} TP_0 \to THH_j$ natural projection.

Recall the descent SS goes like

$$E^1_{s,t}(?(O_K)) = ?_t(O_K/\mathbb{S}_{W(k)}[z]^{\otimes[-s]})$$
$$\Rightarrow ?_{s+t}(O_K/\mathbb{S}_{W(k)})$$

for $? \in \{THH, TC^-, TP\}$, w/

$$d^1 : E^1_{s,t} \longrightarrow E^1_{s-1,t}$$

E^1-term has Nygaard filtration, and from this filtration we get

$$E^1_{i,j,k}(TC^-(O_K)) = H^i(gr^{2k}(TC^-_j(O_K/\mathbb{S}_{W(k)}[z]^{\otimes[-]})))$$
$$\Rightarrow E^2_{-i,j}(TC^-(O_K))$$

$$E^1_{i,j,k}(TP(O_K)) = H^i(gr^{2k}(TP_j(O_K/\mathbb{S}_{W(k)}[z]^{\otimes[-]})))$$
$$\Rightarrow E^2_{-i,j}(TP(O_K))$$

where gr = graded assoc, and all multiplicative SS w/

$$d^r : E^r_{i,j,k} \longrightarrow E^r_{i+1,j,k+r}$$

In fact.

$$E^1_{i,j,k}(TP(\mathcal{O}_K)) = \begin{cases} 0 & , j \text{ odd} \\ E^1_{-i,2k}(THH(\mathcal{O}_K))\,\sigma^{j/2} & , j \text{ even.} \end{cases}$$

$$TC^-_j(\mathcal{O}_K/\mathbb{S}_{W(k)[z]}{}^{\otimes[-]}) \cong \mathcal{N}^{\geq j}\, TP_j(\mathcal{O}_K/\mathbb{S}_{W(k)[z]}{}^{\otimes[-]}).$$

Thus.

<u>Prop</u> Both descent SS for $TC^-(\mathcal{O}_K/\mathbb{S}_{W(k)})$ and $TP(\mathcal{O}_K/\mathbb{S}_{W(k)})$ collapse at E^2-term.

V. Application : computation in $\mathbb{F}_p$-coefficient.

- <u>Tools</u>

Descent SS for TC :

$$E^1_{i,j}(TC(\mathcal{O}_K);\mathbb{F}_p) \Longrightarrow TC_{i+j}(\mathcal{O}_K;\mathbb{F}_p)$$

w/ multiplicative SS. $k \in \{0,1\}$.

$$\tilde{E}^2_{i,k,j}(TC(\mathcal{O}_K);\mathbb{F}_p) \Rightarrow E^2_{i-k,j}(TC(\mathcal{O}_K);\mathbb{F}_p)$$

where

$$\tilde{E}^2_{i,0,j} = \ker(can-\varphi : E^2_{i,j}(TC^-(\mathcal{O}_K);\mathbb{F}_p) \longrightarrow E^2_{i,j}(TP(\mathcal{O}_K);\mathbb{F}_p))$$

$$\tilde{E}^2_{i,1,j} = \operatorname{coker}(can-\varphi : E^2_{i,j}(TC^-(\mathcal{O}_K);\mathbb{F}_p) \longrightarrow E^2_{i,j}(TP(\mathcal{O}_K);\mathbb{F}_p))$$

In order to get E^2-terms, we need :

1) $E^2(THH(\mathcal{O}_K);\mathbb{F}_p)$. which is obtained by cobar cpx assoc. to Hopf algebroid (A,Γ), where

$$A = THH_*(\mathcal{O}_K/\mathbb{S}_{W(k)}[z];\mathbb{F}_p)$$

$$\Gamma = THH_*(\mathcal{O}_K/\mathbb{S}_{W(k)}[z_0,z_1];\mathbb{F}_p)$$

2) Refined version of algebraic Tate SS / HFPSS, from Nygaard filtration on E^1-terms of descent SS for TC^- and TP. The assoc. graded are not hard to compute. They converge to E^2-terms of descent SS.

- Compute E^2-term of $THH(\mathcal{O}_K;\mathbb{F}_p)$.

Again, (A,Γ) forms a Hopf algebroid, and

$$A = \mathcal{O}_K[u]\otimes_{\mathbb{Z}}\mathbb{F}_p = (\mathcal{O}_K/(p))[u]$$
$$= k[z]/(z^{e_K})[u]$$
$$\Gamma = (\mathcal{O}_K\langle t_{z_0-z_1}\rangle \otimes_{\mathcal{O}_K} \mathcal{O}_K[u_0])\otimes_{\mathbb{Z}}\mathbb{F}_p$$
$$= (k[z]/(z_1^{e_K}))[u_0]\langle t_{z_0-z_1}\rangle$$

where z corresponds to $\overline{\pi_K} = (-1)^{\deg \pi_K}\cdot \pi_K$.

<u>Notation</u> μ = leading coeff of $E_K(z)$.

$\tilde{\mu} = -\mu^p/\delta(E_K(z_0))$. δ from δ-ring str. on TP_0.

▲ <u>Thm 1</u>

1) The k-v.s. $E^2_{0,*}(THH(\mathcal{O}_K);\mathbb{F}_p)$ has a basis by

$$\begin{cases} z^{\ell} u^{n} . & 1 \le \ell \le e_K - 1 \text{ or } p \mid e_K n . \text{ for } e_K > 1 \\ u^{n} . & p \mid n . \text{ for } e_K = 1 . \end{cases}$$

2) The k-v.s. $E^{2}_{-1,*}(THH(\mathcal{O}_K); \mathbb{F}_p)$ has a basis by

$$\begin{cases} z_1^{\ell}\left(u_0^{n-1} t_{z_0 - z_1} - (n-1) E_K'(z_0) u_0^{n-2} t^{[2]}_{z_0 - z_1} \right) . & 0 \le \ell \le e_K - 2 \text{ or } \\ & p \mid e_K n . \text{ for } e_K > 1 . \\ \sum_{j=1}^{\ell} \frac{(n-1)!}{(n-j)!} (-\bar{\mu})^{j} u_0^{n-j} t^{[j]}_{z_0 - z_1} . & p \mid n . \text{ for } e_K = 1 \end{cases}$$

3) For $i \neq -1 . 0 .$ $E^{2}_{i,*} = 0 .$

pf sketch. cobar cpx looks like

$$0 \to A \xrightarrow{(D_0 \circ \eta_R) dz} A dz \to 0$$

$\rightsquigarrow$

$$0 \to k[z]/(z^{e_K})[u] \xrightarrow{f(u) \mapsto -e_K \bar{\mu} z^{e_K - 1} f'(u) dz} k[z]/(z^{e_K})[u] dz \to 0$$

We implicitly use the fact that

$$\mathrm{Hom}_A(M . N) = \mathrm{Hom}_{\Gamma}(M . \Gamma \otimes_A N)$$

If we want to get rid of dz in H^1. we need to consider the comm. diagram

$$\begin{tikzcd} A \arrow[r, "(D_0 \circ \eta_R) dz"] \arrow[d, "id"'] & A \arrow[d, "\beta"] \\ A \arrow[r, "\eta_L - \eta_R"] & \Gamma \end{tikzcd}$$

where $\beta : u^n dz \longmapsto \overline{u^{(n+1)}}$ for

$$u^{(n)} = \sum_{j=1}^{n} \frac{(n-1)!}{(n-j)!} (-E_K'(z))^{j-1} u_0^{n-j} t^{[j]}_{z_0 - z_1}$$

Rk The complication arises from the even degree maps. So we modify the Nygaard filtration.

- Hopf algebroid for TP_0.

Refine $\mathcal{N}^{\geq \bullet}$ by indexing over $\frac{2}{e_K}\mathbb{Z}$, i.e. for $TP_0(\mathcal{O}_K/\mathbb{S}_{W(k)}[z]; \mathbb{F}_p)$-mod w/ ordinary Nyg. filtration, the new one is given by

$$\mathcal{N}^{\geq j+m/e_K} M = z^m \mathcal{N}^{\geq j} M + \mathcal{N}^{\geq j+1} M$$

and all are even filtrations. The refined Nyg. fil. gets all what we like. Now write

$$A_0 := TP_0(\mathcal{O}_K/\mathbb{S}_{W(k)}[z]; \mathbb{F}_p)$$

$$\Gamma_0 := TP_0(\mathcal{O}_K/\mathbb{S}_{W(k)}[z_0, z_1]; \mathbb{F}_p)$$

<u>Prop</u> (A_0, Γ_0) Hopf algebroid, w/

$$(A_0, \Gamma_0) = (k[z], k[z_0] \otimes_k k\langle t_{z_0 - z_1}\rangle) \quad \text{s.t.}$$

1) $\eta_L(z) = z_0$, $\eta_R(z) = z_1$, $\varepsilon(z_i) = z$ for $i = 0, 1$.

2) $\Delta(t^{[i]}_{z_0 - z_1}) = \sum_{0 \leq j \leq i} t^{[j]}_{z_0 - z_1} \otimes t^{[i-j]}_{z_0 - z_1}$

$\varepsilon(t^{[i]}_{z_0 - z_1}) = 0$

3) $e_K = 1$: $z_1 = z_0 - t_{z_0 - z_1}$

$e_K > 1$: $z_1 = z_0$. Hopf algebroid = Hopf alg.

- Refined algebraic Tate SS.

Refined Nyg. filtrations on E^1-term of descent SS for TC^- and TP give rise to

$$\tilde{E}^{1/e_K}_{i,j,k}(TP) = H^i(Gr^k(TP_j(\mathcal{O}_K/\mathbb{S}_{W(k)}[z]^{\otimes[-]}))) \Rightarrow E^2_{-i,j}(TP)$$
$$\tilde{E}^{1/e_K}_{i,j,k}(TC^-) = H^i(Gr^k(TC^-_j(\mathcal{O}_K/\mathbb{S}_{W(k)}[z]^{\otimes[-]}))) \Rightarrow E^2_{-i,j}(TC^-)$$

where $k \in 1/e_K\,\mathbb{Z}$. $\forall\, r \in 1/e_K\,\mathbb{Z}_{\geq 1}$

$$d^r : \tilde{E}^r_{i,j,k} \longrightarrow \tilde{E}^r_{i+1,j,k+r}$$

As stated in <u>PART IV</u>, $\tilde{E}^{1/e_K}(TP)$ can be identified w/ cobar cpx for $k[z][\sigma^{\pm}]$ w.r.t. Hopf algebroid (A_0, Γ_0). $\tilde{E}^{1/e_K}(TC^-)$ is just a truncation of $\tilde{E}^{1/e_K}(TP)$.

▲ <u>Thm 2</u>

Write $t_{z_0-z_1}$ as dz. When $e_K = 1$, write

$$\sum_{j=1}^{n} \frac{(n-1)!}{(n-j)!}(-1)^j z_0^{n-j}\, t^{[j]}_{z_0-z_1}$$

which is formally equal to $\frac{1}{n}(z_0^n - z_1^n)$, as $z_0^{n-1}dz$. Then

1) $e_K > 1$.

$$\tilde{E}^{1-1/e_K}_{*,j,*}(TP) = k[z]\sigma^j \oplus k[z_0]\sigma^j dz$$
$$d^{1-1/e_K}(z\sigma^j) = \sigma^j dz$$

2) $e_K = 1$.

$$\tilde{E}^{1-1/e_K}_{*,j,*}(TP) = k[z^p]\sigma^j \oplus z_0^{p-1}k[z_0^p]\sigma^j dz$$

3) The Tate differentials go like :

For $n \geq 0$, $j \in \mathbb{Z}$, $\ell = \nu_p(n - \frac{pe_K j}{p-1})$, and $n' \equiv p^{-\ell}(n - \frac{pe_K j}{p-1}) \mod p$, we have

$$d^{\frac{p^{\ell+1}-1}{p-1} - \frac{1}{e_K}}(z^n\sigma^j) = n'\,\overline{\tilde{\mu}}^{\frac{p^\ell - 1}{p-1}} z_0^{pe_K\cdot\frac{p^\ell-1}{p-1}+n-1}\sigma^j dz$$

These are the only non-trivial differentials. Here $v_p(-)$ is the p-adic valuation = biggest integer e s.t. $p^e \mid -$.

pf IDEA: 1), 2) reduce to TP_0 and by cobar cpx.

3) by δ-ring str on TP_0. and see how the elets go in the $(p.\mathcal{N})$-top (or $\mathcal{N}$-top). Finally by comparing w/ part (1) & (2) to determine the differentials.

- Compute $E^2(TC^-)$ and $E^2(TP)$.

▲ <u>Thm 3</u>

1) For $j \in \mathbb{Z}$.

$$E^2_{0,2j}(TP) = \begin{cases} k\{\text{cycle w/ leading term } z^{\frac{pe_K j}{p-1}} \sigma^j\}. & p-1 \mid e_K j \text{ and } j \geq 0 \\ 0. & \text{else}. \end{cases}$$

and $\text{can}: E^2_{0,*}(TC^-) \longrightarrow E^2_{0,*}(TP)$ is iso.

2) $E^2_{-1,2j}(TP)$ is a k-v.s. w/ basis

① $z_0^{\frac{pe_K\cdot(j-1)+bp^\ell}{p-1}-1}\sigma^j dz$. w/ $\ell \geq 1$. $b \in \mathbb{Z}$ s.t.

$$-\frac{1}{p^{\ell-1}} e_K\cdot(j-1) < b < pe_K - \frac{e_K j}{p^{\ell-1}}. \quad p \nmid b.$$

$$b \equiv -e_K(j-1) \mod p-1$$

② $z_0^{\frac{pe_K\cdot(j-1)}{p-1}-1}\sigma^j dz$. if $j>1$ and $p-1 \mid e_K\cdot(j-1)$.

3) For $j \geq 1$. $E^2_{-1,2j}(TC^-)$ is a k-v.s. w/ basis

① $z_0^{\frac{pe_K\cdot(j-1)+bp^\ell}{p-1}-1}\sigma^j dz$. w/ $\ell \geq 0$. $b \in \mathbb{Z}$ s.t.

$$-\frac{1}{p^{\ell}} e_K\cdot(j-1) < b < pe_K - \frac{e_K j}{p^{\ell}}. \quad p \nmid b.$$

$b \equiv -e_K(j-1) \mod p-1$

② $z_0^{\frac{pe_K\cdot(j-1)}{p-1}-1}\sigma^j dz$. if $j>1$ and $p-1 \mid e_K\cdot(j-1)$.

4) For $j \geq 1$,

$\ker(\mathrm{can}: E^2_{-1,2j}(TC^-) \longrightarrow E^2_{-1,2j}(TP))$

is an $e_K\cdot j$-dim k-v.s. w/ basis w/ leading terms

$$z_0^{\frac{pe_K\cdot(j-1)+bp^{\ell}}{p-1}-1}\sigma^j dz$$

w/ $\ell \geq 0$. $b \in \mathbb{Z}$ s.t.

$p \nmid b$. $b \equiv -e_K\cdot(j-1) \mod p-1$

$$p^{\ell-1} \leq \frac{e_K j}{pe_K - b} < p^{\ell}.$$

- Compute the cyclotomic Frobenius

<u>Prop</u> For $n \geq e_K\cdot j$.

$$\varphi(z^n\sigma^j) = \bar{\mu}^{-pj} z^{p(n-e_K j)}\sigma^j$$

$$\begin{aligned}\text{LHS} &= \varphi(z^{n-e_K j})\,\bar{\mu}^{-pj}\,\varphi(E_K(z)\sigma)^j \\ &= \bar{\mu}^{-pj} z^{p(n-e_K\cdot j)}\,\varphi(u)^j \\ &= \bar{\mu}^{-pj} z^{p(n-e_K\cdot j)}\sigma^j\end{aligned}$$

This tells us how φ on $E^2_{0,*}(-)$ behaves.

<u>Prop</u> 1) For $j \geq 1$. if $\alpha \in E^2_{-1,2j}(TC^-)$ detected by $z_0^{n-1}\sigma^j dz$

then $\varphi(\alpha) \in E^2_{-1,2j}(TP)$ is detected by

$$-\bar{\mu}^{-p(j-1)} z_0^{p(n-e_K\cdot(j-1))-1}\sigma^j dz$$

Thus, $\varphi: E^2_{-1,2j}(TC^-) \longrightarrow E^2_{-1,2j}(TP)$ is iso for $j \geq 1$.

2) Suppose $\alpha \in E^2_{-1,2j}\, TC^-$ has refined Nygaard filtration m. Then

① If $j \leq 0$. $\varphi(\alpha)$ is $> m$

② If $j \geq 1$. $\varphi(\alpha)$ is $>/</=$ m iff

$$m >/</= \frac{p^j - 1}{p-1} - \frac{1}{e_K}.$$

To see how the canonical does. we have

<u>Prop</u> 1) $j > 0$. $\mathrm{can}: E^2_{-1,2j}(TC^-) \longrightarrow \mathcal{N}^{\geq j} E^2_{-1,2j}(TP)$

surjective.

2) $j \leq 0$. $\mathrm{can}: E^2_{-1,2j}(TC^-) \longrightarrow E^2_{-1,2j}(TP)$

iso.

3) $m \geq j$. $\mathrm{can}: \mathcal{N}^{\geq m} E^2_{-1,2j}(TC^-) \longrightarrow \mathcal{N}^{\geq m} E^2_{-1,2j}(TP)$

surjective.

pf. Follows from the corresponding result on

$\mathrm{can}: TC^-_{2j}(\mathcal{O}_K / \mathbb{S}_{W(k)}[z]^{\otimes[-]}) \to TP_{2j}(\mathcal{O}_K / \mathbb{S}_{W(k)}[z]^{\otimes[-]})$

<u>Cor</u> 1) $j \geq 1$. $m \geq \frac{p^j - 1}{p-1}$. then

$\mathrm{can} - \varphi: \mathcal{N}^{\geq m} E^2_{-1,2j}\, TC^- \longrightarrow \mathcal{N}^{\geq m} E^2_{-1,2j}\, TP$

is surjective

2) $j \leq 0$. then

$\mathrm{can} - \varphi: E^2_{-1,2j}\, TC^- \longrightarrow E^2_{-1,2j}\, TP$ iso.

Now we can compute the E^2-term of descent SS for TC.

Let β be the elet in $\tilde{E}^2_{0,0,2d}\,TC \subseteq E^2_{0,2d}\,TC^-$ detected by $\bar{\mu}^{\frac{pd}{p-1}} z^{\frac{pe_k d}{p-1}} \sigma^d$, where d is the minimal number s.t.

$$p-1 \mid e_k d, \quad N_{k|\mathbb{F}_p}(\bar{\mu})^d = 1,$$

and $N_{k|\mathbb{F}_p} : k \longrightarrow \mathbb{F}_p$ is the norm.

Since $\tilde{E}^2_{i,k,j}\,TC \Rightarrow E^2_{i-k,j}\,TC$, $k = 0, 1$, we only have to consider the case $i = 0, -1$, or equivalently, just to compute $\tilde{E}^2_{0,0,*}$, $\tilde{E}^2_{0,1,*}$, $\tilde{E}^2_{-1,1,*}$, $\tilde{E}^2_{-1,0,*}$.

To compute:

1° $\tilde{E}^2_{0,0,*} = \mathbb{F}_p[\beta]$.

By Theorem 3, $E^2_{0,2j} = k\{z^{\frac{pe_k j}{p-1}}\sigma^j\}$ when $p-1 \mid e_k j$ and $j \geq 0$. Now, $\varphi(z^{\frac{pe_k j}{p-1}}\sigma^j) = \bar{\mu}^{-pj} z^{\frac{pe_k j}{p-1}}\sigma^j$.

So $\lambda z^{\frac{pe_k j}{p-1}}\sigma^j \in \tilde{E}^2_{0,0,*} \iff$

$$(can - \varphi)(\lambda z^{\frac{pe_k j}{p-1}}\sigma^j) = \lambda z^{\frac{pe_k j}{p-1}}\sigma^j - \varphi(\lambda)\cdot\bar{\mu}^{-pj} z^{\frac{pe_k j}{p-1}}\sigma^j$$
$$= 0$$

$$\iff \bar{\mu}^{pj} = \lambda^{-1}\varphi(\lambda) = \lambda^{p-1} \text{ for some } \lambda \in k$$

So $d \mid j$ by $N_{k|\mathbb{F}_p}(\bar{\mu})^j = 1$. Conversely, if $d \mid j$, then $\lambda = \lambda' \bar{\mu}^{\frac{pd}{p-1}}$ w/ $\lambda' \in k$. We get the result.

2° $\tilde{E}^2_{0,1,*}$ is a free rk 1 $\mathbb{F}_p[\beta]$-module.

Similarly. Note $can = iso$, $\varphi(z^n\sigma^j) = \bar{\mu}^{-pj} z^{p(n-e_k j)}\sigma^j$.

$\tilde{E}^2_{0,1,*} = E^2_{0,*}(TP)/(can - \varphi)$.

3° $\tilde{E}^2_{-1,1,*} = \mathbb{F}_p[\beta]\{\mathrm{can}\ \gamma\}$. where $\gamma \in \ker(\mathrm{can}-\varphi)$ detected by $\bar{\mu}^{\frac{pd}{p-1}} z^{\frac{pe_K d}{p-1}-1}\sigma^{d+1}dz$

1) Suppose $\gamma_0 \in E^2_{-1,2d+2}TC^-$ is detected by $z^{\frac{pe_K d}{p-1}-1}\sigma^{d+1}dz$

Then $\varphi(\gamma_0)$ is detected by $\bar{\mu}^{-pd} z^{\frac{pe_K d}{p-1}-1}\sigma^{d+1}dz$. It follows that $(\mathrm{can}-\varphi)(\bar{\mu}^{\frac{pd}{p-1}}\gamma_0) \in \mathcal{N}^{\geq \frac{pd}{p-1}+1} E^2_{-1,2d+2} TP$

and $\mathrm{can}-\varphi$ is surjective in this case. hence γ is defined.

2) $\forall\ \alpha \in \mathrm{coker}(\mathrm{can}-\varphi)$ w/ highest leading term. Now $\alpha \in E^2_{-1,2j}TC$. $j \geq 1$ and leading deg $\alpha \in [1, \frac{pj-1}{p-1}-\frac{1}{e_K}]$

On the other hand, if leading deg α not in this interval then $\exists\ \alpha'$ s.t. $\varphi(\alpha') = \alpha$. α' has higher leading deg $\geq \alpha$. but $\mathrm{can}\ \alpha' = \alpha$ iso. contradiction.

Thus α has leading term $\frac{pj-1}{p-1}-\frac{1}{e_K}$ $\Rightarrow$ it is detected by some $\lambda z_1^{\frac{pe_K(j-1)}{p-1}-1}\sigma^j dz$. Thus $d \mid j-1$.

$\alpha \in \mathbb{F}_p[\beta]\{\mathrm{can}\ \gamma\}$.

4° $\tilde{E}^2_{-1,0,*} = \mathbb{F}_p[\beta]\{\gamma, \tilde{\alpha}_\bullet\}$. where $\tilde{\alpha}_\bullet$ is a family of cycles detected by $c z^{\frac{pe_K(j-1)+bp^\ell}{p-1}-1}\sigma^j dz \in E^2_{-1,2j}TC$.

w/ $\ell \geq 0$ and $0 < b < pe_K$. $p \nmid b$. $1 \leq j \leq d$.

$b \equiv -e_K(j-1) \mod p-1$. $p^{\ell-1} \leq \frac{e_K j}{pe_K - b} < p^\ell$. and

c runs over a basis of k over $\mathbb{F}_p$.

$j \leq 0$: $\ker(\mathrm{can}-\varphi)$ trivial.

$j \geq 1$: leading deg $\alpha \in \tilde{E}^2_{-1,0,*}TC$ is $\geq \frac{pj-1}{p-1}-\frac{1}{e_K}$.

If "=". then $\exists\ \beta' \in \mathbb{F}_p[\beta]\{\gamma\}$ s.t. $\alpha-\beta'$ satisfies leading deg $> \frac{p^j-1}{p-1} - \frac{1}{e_K}$.

If "$>$", then

$$\mathrm{can}: \mathcal{N}^{> \frac{p^j-1}{p-1} - \frac{1}{e_K}} E^2_{-1,2j} TC^- \to \mathcal{N}^{> \frac{p^j-1}{p-1} - \frac{1}{e_K}} E^2_{-1,2j} TP$$

surjective, cocycles in the statement of Prop 4° form an $\mathbb{F}_p$-basis of $\ker(\mathrm{can})$. The remaining cycles in $\mathcal{N}^{> \frac{p^j-1}{p-1} - \frac{1}{e_K}}$ (denoted S) induces an iso to $\mathcal{N}^{> \frac{p^j-1}{p-1} - \frac{1}{e_K}} E^2_{-1,2j} TP$. Write $\alpha = \alpha_1 + \alpha_2$, $\alpha_1 \in \ker(\mathrm{can})$, $\alpha_2 \in S$. Then

$$(\mathrm{can} - \varphi)(\alpha_2) = \varphi(\alpha_1)$$

φ raises filtration $\Rightarrow$

$$\alpha_2 = (1 - \mathrm{can}^{-1}\varphi)^{-1}(\mathrm{can}^{-1}\varphi(\alpha_1))$$
$$= \textstyle\sum_{i \geq 1} (\mathrm{can}^{-1}\varphi)^i(\alpha_1)$$

Thus $\alpha \mapsto \alpha_1$ induces iso between $\ker(\mathrm{can})$ and $\ker(\mathrm{can} - \varphi)$ preserving the leading term.

Combine 1° ~ 4°, we have

Thm As $\mathbb{F}_p[\beta]$-mod.

$$E^2_{0,*}(TC(\mathcal{O}_K); \mathbb{F}_p) = \mathbb{F}_p[\beta]$$

$$E^2_{-1,*}(TC(\mathcal{O}_K); \mathbb{F}_p) = \mathbb{F}_p[\beta]\{\lambda\} \oplus \mathbb{F}_p[\beta]\{\alpha^{(j)}_{i,\ell} : 1 \leq i \leq e_K,\ 1 \leq j \leq d,\ 1 \leq \ell \leq f_K\}$$

$$E^2_{-2,*}(TC(\mathcal{O}_K); \mathbb{F}_p) = \mathbb{F}_p[\beta]\{\lambda\gamma\}$$

w/ $|\lambda| = (-1, 0)$, $|\gamma| = (-1, 2(d+1))$, $|\alpha^{(j)}_{i,\ell}| = (-1, 2j)$.

and $\forall\ i \neq 0, -1, -2$,

$$E^2_{i,*}(TC(\mathcal{O}_K); \mathbb{F}_p) = 0.$$

By deg reason, the descent SS collapses at E^2-term. For p odd, descent SS is multiplicative, so no hidden extensions. We have:

Thm p odd, as $\mathbb{F}_p[\beta]$-mod,

$$TC_*(\mathcal{O}_K; \mathbb{F}_p) = \mathbb{F}_p[\beta]\{1, \lambda, \gamma, \lambda\gamma\} \oplus \mathbb{F}_p[\beta]\{\alpha^{(j)}_{i,\ell} : 1 \le i \le e_K,\ 1 \le j \le d,\ 1 \le \ell \le f_K\}$$

w/ $|\beta| = 2d$, $|\lambda| = -1$, $|\gamma| = 2d+1$, $|\alpha^{(j)}_{i,\ell}| = 2j-1$.

Actually, $d = [K(\zeta_p) : K]$.

Thm $p = 2$ (descent SS not multiplicative), then as a $\mathbb{Z}_2[\beta^4]$-mod,

$$TC_*(\mathcal{O}_K; \mathbb{F}_2) = \begin{cases} \mathbb{F}_2[\beta^2]\{1\} \oplus \mathbb{Z}/4[\beta^2]\{\beta\} \oplus \mathbb{F}_2[\beta]\{\lambda, \gamma\} \oplus \mathbb{F}_2[\beta^2]\{\beta\lambda\gamma\} \oplus \mathbb{F}_2[\beta]\{\alpha_{i,\ell} : 1 \le i \le e_K,\ 1 \le \ell \le f_K\}, & \text{if } [K : \mathbb{Q}_2] \text{ odd.} \\ \mathbb{F}_2[\beta]\{1, \lambda, \gamma, \lambda\gamma\} \oplus \mathbb{F}_2[\beta]\{\alpha_{i,\ell} : 1 \le i \le e_K,\ 1 \le \ell \le f_K\}, & \text{if } [K : \mathbb{Q}_2] \text{ even.} \end{cases}$$

w/ $|\beta| = 2$, $|\lambda| = -1$, $|\gamma| = 3$, $|\alpha_i| = 2i-1$.